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The primitive permutation groups of certain degrees

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Abstract

This paper precisely classifies all simple groups with subgroups of index n and all primitive permutation groups of degree n, where $n = 2{\cdot}3^r$, $5{\cdot}3^r$ or $10{\cdot}3^r$ for $r \geq 1$. As an application, it proves positively Gardiner and Praeger's conjecture in [6] regarding transitive groups with bounded movement.

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1. Introduction

Let Ω be a finite set of n elements and G a transitive permutation group on Ω. Let $H = G_\alpha$ for some $\alpha \in \Omega$. Then $|\Omega| = |G : H| = n$. Determining primitive permutation groups with given degrees n is a long-standing problem in permutation group theory. For certain values of n, the primitive permutation groups G of degrees n have been classified; for example, Sims [26] for $n \leq 20$, Tan and Wang [28] for $21 \leq n \leq 30$, Dixon and Mortimer [5] for $n \leq 1000$; Guralnick [8] for G simple and n a power of a prime, and Liebeck and Saxl [20] for n odd and [21] for $n = mp$ with $m < p$ and p prime. In this paper, using the classification of finite simple groups, we precisely classify all simple groups with subgroups of index n and all primitive permutation groups of degree n where $n = 2{\cdot}3^r$, $5{\cdot}3^r$ and $10{\cdot}3^r$ for $r \geq 1$.

However, the main motivation of this work came from a conjecture of Gardiner and Praeger regarding transitive groups with bounded movement defined as follows. A

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transitive G is said to have "bounded movement" condition:

$$\mathrm{BM}(m)\colon \text{ if } g \in G \text{ and } \Delta \subseteq \Omega \text{ satisfy } \Delta \cap \Delta^g = \emptyset, \text{ then } |\Delta| \le m.$$

Every permutation group G satisfies $\mathrm{BM}(m)$ for some smallest $m \le |\Omega|/2$. On the other hand, if G satisfies $\mathrm{BM}(m)$ for some m, then Praeger [24] proved $|\Omega| \le 3m$. This simple bound is sharp and Gardiner and Praeger [6] posed

Conjecture. *Let G be a transitive permutation group of degree $3m$ satisfying $\mathrm{BM}(m)$. Then either $G = S_3$ and $m = 1$, or $G = A_4$ or A_5 and $m = 2$, or G is a 3-group.*

About this conjecture, Praeger [24] proved $3m = 3^r$, $2{\cdot}3^r$, $5{\cdot}3^r$ or $10{\cdot}3^r$, and Gardiner and Praeger [6] proved that the minimal counterexample to this conjecture is a simple group acting primitively on Ω, where $|\Omega| = 3m$. We shall show that this conjecture is true by checking our list of the simple groups with subgroups of the corresponding indexes.

Table 1

G	H	$\|G:H\|$	H is maximal?
A_5	D_{10}	6	Yes
	Z_2^2	15	No
	Z_2	30	No
A_6	S_4	15	Yes
	A_4	30	No
	D_8	45	No
	Z_2^2	90	No
A_7	$L_2(7)$	15	Yes
A_8	$2^3{\cdot}L_3(2)$	15	Yes
A_{10}	S_8	45	Yes
	A_8	90	No
A_c	A_{c-1}	$3^r{\cdot}l$	Yes
$L_2(8)$	$2^3{\cdot}Z_7$	9	Yes
$U_4(2)$	$2^4{\cdot}A_5$	27	Yes
	$2^4{\cdot}A_4$	$3^3{\cdot}5$	No
	$2^4{\cdot}D_{10}$	$3^4{\cdot}2$	No
	$2^4{\cdot}2^2$	$3^4{\cdot}5$	No
	$2^4{\cdot}D_6$	$3^3{\cdot}10$	No
	$2^4{\cdot}Z_2$	$3^4{\cdot}10$	No
	$2{\cdot}(A_4 \times A_4){\cdot}2$	45	Yes
	$2{\cdot}(A_4 \times A_4)$	90	No
	$2(2^2 \times A_4)$	$3^3{\cdot}10$	No
$U_4(3)$	$L_3(4)$	$3^4{\cdot}2$	Yes
$Sp_6(2)$	$2^6{\cdot}L_3(2)$	$3^3{\cdot}5$	Yes
$P\Omega_8^+(2)$	$2^6{\cdot}A_8$	$3^3{\cdot}5$	Yes
$L_2(p)$	$Z_p{\cdot}Z_{(p-1)/2}$	$3^r{\cdot}2$ or $3^r{\cdot}10$	Yes
	$Z_p{\cdot}Z_{(p-1)/10}$	$3^r{\cdot}10$	No
$L_2(p^2)$	$Z_p^2{\cdot}Z_{(p^2-1)/2}$	$3^r{\cdot}10$	Yes

Table 2

T	$H \cap T$	n
A_5	D_{10}	6
A_6	S_4	15
A_7	$L_2(7)$	15
A_8	$2^3 \cdot L_3(2)$	15
A_{10}	S_8	45
A_c	A_{c-1}	$3^r \cdot l$
$L_2(8)$	$2^3 \cdot Z_7$	9
$U_4(2)$	$2^4 \cdot A_5$	27
	$2 \cdot (A_4 \times A_4) \cdot 2$	45
$U_4(3)$	$L_3(4)$	$3^4 \cdot 2$
$Sp_6(2)$	$2^6 \cdot L_3(2)$	$3^3 \cdot 5$
$P\Omega_8^+(2)$	$2^6 \cdot A_8$	$3^3 \cdot 5$
$L_2(p)$	$Z_p \cdot Z_{(p-1)/2}$	$3^r \cdot 2$ *or* $3^r \cdot 10$
$L_2(p^2)$	$Z_p^2 \cdot Z_{(p^2-1)/2}$	$3^r \cdot 10$

The main results of this paper are the following theorems:

Theorem 1.1. *Let G be a nonabelian simple group and H a subgroup of G such that $|G : H| = 3^r l$, where $r \geq 1$ and $l \mid 10$. Then G, H and $|G : H|$ are listed in Table 1.*

Theorem 1.2. *Let G be a primitive permutation group of degree $n = 3^r l$ where $r \geq 1$ and $l \mid 10$ on a set Ω and let $H = G_\alpha$, $\alpha \in \Omega$. Then one of the following is true:*

(i) *$Z_3^d \lhd G \leq AGL(d,3)$ for some integer $d \geq 1$;*

(ii) *G, H and n are listed as follows:*

G	H	n	*Remark*
$L_2(8) \wr P$	$(2^3 \cdot Z_7) \wr P$	3^{2m}	
$U_4(2) \wr P$	$(2^4 \cdot A_5) \wr P$	3^{3m}	
$A_c \wr P$	$A_{c-1} \wr P$	3^{tm}	$c = 3^m$ and $t \geq 2$

where $m \geq 2$ and P is a transitive group of degree m;

(iii) *$soc(G) = T$ is simple and one of the items in Table 2 holds:*

Remark. By Theorem 1.2, we know that if G is a primitive permutation group of degree $3^r 2$, $3^r 5$ or $3^r 10$ for $r \geq 1$ then either G is 2-transitive or $(G, G_\alpha) = (A_6, S_4)$, (A_{10}, S_8), $(U_4(2), 2^4 \cdot A_5)$, $(U_4(2), 2 \cdot (A_4 \times A_4) \cdot 2)$, $(U_4(3), L_3(4))$, $(Sp_6(2), 2^6 \cdot L_3(2))$ or $(P\Omega_8^+(2), 2^6 \cdot A_8)$.

The third theorem shows that Gardiner and Praeger's conjecture is true:

Theorem 1.3. *Let G be a transitive permutation group of degree $3m$ satisfying* BM(m). *Then one of the following holds*:
(a) $G = S_3$, $m = 1$;
(b) $G = A_4, A_5$, $m = 2$;
(c) G *is a* 3-*group.*

The notation and terminology used in this paper are standed (see [3, 14, 27]). In particular, for an integer n, $n|_p$ and $n|_{p'}$ denote the p-part and the p'-part of n, respectively.

2. Preliminaries

This section quotes some preliminary results used in the following sections.

Lemma 2.1. *Suppose that p is a prime and $n \geq 3$. Then*
(1) ([9, IX 8.3 and 8.4]) *there exists a prime $k > n$ such that $k \mid p^n - 1$ and $k \nmid p^i - 1$ for all $0 < i < n$, except in the case $p = 2$ and $n = 6$;*
(2) *if $n = 2m$ is even, then there is prime $k > n$ such that $k \mid p^m + 1$ but $k \nmid p^i - 1$ for all $i < n$, except in the case $p = 2$ and $m = 3$.*

Proof. Clearly part (2) is an immediate consequence of part (1). □

Lemma 2.2 ([8, 2.1]). *Let $C(n,k)$ denote the binomial coefficient $n!/k!(n-k)!$. If $C(n,k) = p^a m$, where $(p,m) = 1$, then $p^a \leq n$. For a natural number, there is a prime k such that $n < k < 2n$.*

Lemma 2.3. *Let G be a finite simple group and $\pi(|G|)$ the set of all prime divisors of $|G|$. Then (where the number after G is the order of G)*
(1) ([7, pp. 12–14]) *if $|\pi(|G|)| = 3$ then $G = A_5$ $(2^2 3 \cdot 5)$, A_6 $(2^3 3^2 5)$, $L_2(7)$ $(2^3 3 \cdot 7)$, $L_2(8)$ $(2^3 3^2 7)$, $L_2(17)$ $(2^4 3^2 17)$, $L_3(3)$ $(2^4 3^3 13)$, $U_3(3)$ $(2^5 3^3 7)$ or $U_4(2)$ $(2^6 3^4 5)$;*
(2) ([25]) *if $\pi(|G|) = \{2,3,7,p\}$ and $7^2 \nmid |G|$, then $G = A_7$ $(2^3 3^2 5 \cdot 7)$, A_8 $(2^6 3^2 5 \cdot 7)$, A_9 $(2^6 3^4 5 \cdot 7)$, A_{10} $(2^7 3^4 5^2 7)$, $L_2(13)$ $(2^2 3 \cdot 7 \cdot 13)$, $L_2(27)$ $(2^2 3^3 7 \cdot 13)$, $L_2(127)$ $(2^7 3^2 7 \cdot 127)$, $L_3(4)$ $(2^6 3^2 5 \cdot 7)$, $Sp_6(2)$ $(2^9 3^4 5 \cdot 7)$, $O_8^+(2)$ $(2^{12} 3^5 5^2 7)$, $G_2(3)$ $(2^6 3^6 7 \cdot 13)$, $U_3(5)$ $(2^4 3^2 5^3 7)$ or $U_4(3)$ $(2^7 3^6 5 \cdot 7)$.*

3. The simple groups with a subgroup of index $3^r l$

In this section, we prove Theorem 1.1. Let G be a simple group and $H < G$. If $|G : H| = 3^r l$ for some $l \mid 10$, then there is a maximal subgroup M of G such that

$|G:M|\,\big|\,3^r l$. Thus we suppose that G has a maximal subgroup of index $3^r l$. We shall check the four types of simple groups separately as follows.

3.1. The alternating groups and the sporadic simple groups

Lemma 3.1. *If G is an alternating simple group and H is a subgroup of G of index $3^r l$ where $r \geq 1$ and $l\,\big|\,10$, then Theorem 1.1 holds.*

Proof. Let $G = A_n$, the alternating group of degree n, and H a maximal subgroup of G of index $3^r l$. Let Ω be a set of n-elements and let G act naturally on Ω.

First assume that H is not transitive on Ω. Then it is easy to see $H = (S_m \times S_{n-m}) \cap G = (A_m \times A_{n-m}) \cdot 2$ with $m < n/2$. Thus $|G:H| = n!/m!(n-m)!$. By Lemma 2.2, $3^r \leq n$. Thus

$$\frac{(n-1)!}{m!(n-m)!} = \frac{|G:H|}{n} \leq \frac{|G:H|}{3^r} = l \leq 10.$$

Since $m < n/2$, $n \geq 2m+1$ and $(n-i)/(m-i+1) > (n-1)/m \geq 2$ for $i = 2,\dots,m-1$. Thus

$$10 \geq \frac{(n-1)!}{m!(n-m)!} = \frac{(n-1)(n-2)\cdots(n-m+1)}{m(m-1)\cdots 2} > \left(\frac{n-1}{m}\right)^{m-1} \geq 2^{m-1}.$$

So $m \leq 4$. It is straightforward to check that all possibilities for (G,H) are listed in Theorem 1.1.

Next assume that H is transitive on Ω. We use induction on $n = |\Omega|$. Clearly if $n = 5$, then the lemma is true. Suppose that the lemma is true for all alternating groups of degrees less than n. For any $\alpha \in \Omega$, $G_\alpha = A_{n-1}$. Since H is transitive on Ω, we have $|A_{n-1} : H_\alpha| = |G_\alpha : H_\alpha| = |G:H|$. Thus H_α is a subgroup of A_{n-1} of index $3^r l$. By the induction assumption, (A_{n-1}, H_α) is in the list of Theorem 1.1. If $H_\alpha = A_{n-2}$ then $n \leq |G:H| = |A_{n-1} : A_{n-2}| = n-1$, which is not possible. Thus $H_\alpha \neq A_{n-2}$ and so $n-1 \leq 10$. A straightforward check by the information in the Atlas [3] shows the list of the lemma is complete. $\square$

Lemma 3.2. *If G is a sporadic simple group, then G has no subgroup H of index $3^r l$, where $r \geq 1$ and $l\,\big|\,10$.*

Proof. If $G = Th, J_4, Fi'_{24}$, BM or M, then there is a prime $p > 5$ such that $p^2\,\big|\,|G|$. Since $p \nmid |G:H|$, H contains a Sylow p-subgroup of G. All such subgroups H have been listed in [2]. Using this list, together with the Atlas [3] and [16], it is easy to check $|G:H| \nmid 3^r 10$.

For the other sporadic groups G, if $G = Fi_{23}$ then all maximal subgroups of G are listed in [15]; if $G \neq Fi_{23}$ then all maximal subgroups H of G are listed in the Atlas [3]. It is easy to check that $|G:H| \nmid 3^r \cdot 10$. $\square$

3.2. The exceptional simple groups of Lie type

Let G be an exceptional simple group of Lie type over $GF(q)$ where $q = p^e$. Suppose that H is a maximal subgroup of G. Then by [22], either $|H| < q^{k(G)}$ where $k(G)$ are listed in [22, Table 1], or H is a parabolic subgroup of G, or H is explicitly listed in [22, Table 1]. We shall treat these cases separately. Note that since $3 \nmid |Sz(q)|$, we assume $G \neq Sz(q)$.

Lemma 3.3. *If H is a maximal subgroup of G with $|H| < q^{k(G)}$, then $|G : H| \nmid 3^r \cdot 10$.*

Proof. Let $n_0 = |G|/q^{k(G)}$. Since $|H| < q^{k(G)}$, we have $|G : H| > n_0$. By [22, Table 1], it is easy to obtain G, $q^{k(G)}$ and n_0. A straightforward checking shows that all possibilities are $G = G_2(2)$ and $G_2(3)$. By the Atlas [3], G has no maximal subgroup of index $3^r l$. □

Lemma 3.4. *If H is a parabolic subgroup then $|G : H| \nmid 3^r l$ where $r \geq 1$ and $l \mid 10$.*

Proof. Assume that H is a parabolic subgroup of G with index $3^r l$. If $G = {}^2G_2(q)$, ${}^3D_4(q)$ or ${}^2F_4(q)$, then by [4, 12, 13, 23], all such H have been known. It is easy to check that $|G : H| \nmid 3^r l$. Suppose now $G \neq {}^2G_2(q), {}^2F_4(q)$ or ${}^3D_4(q)$. Since H is a parabolic subgroup, H is an extension of a p-group by the Chevalley group C determined by a maximal subdiagram of the Dynkin diagram of G. It is easy to show that there is a positive integer m such that any prime divisor of $|H|$ divides q or $q^i - 1$ for $i \leq m$ and that there is an $n > m$ such that $n \geq 6$ and $q^n - 1 \mid |G|$ with $q^n \neq 2^6$. By Lemma 2.1, there is a prime $k \mid q^n - 1$ such that $k > n$ and $k \nmid q^i - 1$ for all $i < n$. Thus $k \nmid |H|$ and so $k \mid |G : H|$, a contradiction. □

Lemma 3.5. *Suppose that H is a group listed in* [22, *Table* 1]. *Then $|G : H| \nmid 3^r \cdot 10$.*

Proof. By [22, Table 1], all H have been explicitly listed. Straightforward calculations show that $|G : H| \nmid 3^r 10$. □

3.3. The classical simple groups of Lie type

Let G be a classical simple group of Lie type on $V(n, q)$, where $q = p^e$. Aschbacher [1] divided the maximal subgroups of G into nine collections: $\mathscr{C}_1$–$\mathscr{C}_8$ and $\mathscr{S}$. We shall treat these collections separately. First recall a result of Liebeck [17].

Theorem 3.6 ([17]). *Let G be a classical simple group and H a maximal subgroup of G in collection $\mathscr{S}$. Then one of the following holds*:

(i) $|H| < \begin{cases} q^{2n+4} & \textit{if } G \textit{ is not unitary,} \\ q^{4n+8} & \textit{if } G \textit{ is unitary;} \end{cases}$
(ii) $H = A_c$ *or* S_c, *where* $5 \le c = n+1$ *or* $n+2$;
(iii) $soc(H)$ *and* G *are in*

$soc(H)$	G
$L_d(q)$	$L_{d(d-1)/2}(q)$
$\Omega_7(q)$	$P\Omega_8^+(q)$
$\Omega_9(q)$	$P\Omega_{16}^+(q)$
$P\Omega_{10}^+(q)$	$L_{16}(q)$
$E_6(q)$	$L_{27}(q)$
$E_7(q)$	$PSp_{56}(q), q$ *odd*
	$P\Omega_{56}^+(q), q$ *even*
M_{24}	$L_{11}(2)$
Co_1	$\Omega_{24}^+(2)$

We shall analyze the three cases in inverse order separately.

Lemma 3.7. *If H is a group in Theorem* 3.6(iii), *then* $|G:H| \nmid 3^r \cdot 10$.

Proof. It is easy to know that $4 \mid |L_{11}(2) : M_{24}|$ and $4 \mid |\Omega_{24}^+ : Co_1|$, so the lemma is true for these cases. For the other cases, it is easy to show that $q^2 \mid |G:H|$, so $p = 3$, and it is equally easy to show that $|G:H|_{p'} > 10$, so $|G:H| \nmid 3^r 10$. □

Lemma 3.8. *If H is a group in Theorem* 3.6(ii), *then* $|G:H| \nmid 3^r \cdot 10$.

Proof. Note that now $|H| \mid (n+2)!$ and $n \ge 3$. Assume that $G = L_n(q)$. If $n \ge 9$, then $(q^n-1)(q^{n-1}-1)(q^{n-2}-1) \mid |G|$. By Lemma 2.1, there are primes k_j such that $k_j > n-j$, $k_j \mid q^{n-j}-1$ and $k_j \nmid q^i - 1$ for all $i < n-j$, where $j = 0, 1, 2$. Clearly, $k_j > n-2 \ge 7$. If $|G:H| \mid 3^r 10$, then $k_j \mid |H|$, so there are three different primes among $n+2$, $n+1$, n and $n-1$, which is not possible. Thus $n \le 8, \pi(|H|) \subseteq \{2,3,5,7\}$ and $7^2 \nmid |H|$. If $|G:H| \mid 3^r 10$ then $\pi(|G|) \subseteq \{2,3,5,7\}$ and $7^2 \nmid |G|$. By Lemma 2.3 and the Atlas [3], this is not possible. For the other classical simple groups, similar arguments deduce $|G:H| \nmid 3^r 10$. □

Lemma 3.9. *Suppose $|H| < q^{2n+4}$ for $G \ne U_n(q)$, and $|H| < q^{4n+8}$ for $G = U_n(q)$. If $|G:H| = 3^r l$ for some $r \ge 1$ and some $l \mid 10$ then $(G,H) = (L_2(9), A_5)$ or $(U_4(3), L_3(4))$.*

Proof. Suppose $|G:H| = 3^r l$ for some $l \mid 10$ and some $r \geq 1$. Assume $G = L_n(q)$. If $n \geq 7$ then $n(n-1)/2 > (2n+4)+3$. Since $q^{n(n-1)/2} \mid |G|$, $q^3 \mid |G:H|$ and so $p = 3$. However, $q^i - 1 > q^{i-1}$ for $i \geq 3$ and $q^2 - 1 > q(n, q-1)$. So

$$|G:H|_{p'} > \frac{(q^n-1)\cdots(q^3-1)(q^2-1)/(n,q-1)}{q^{2n+4}} > \frac{q^{(n-1)+\cdots+2+1}}{q^{2n+4}}$$
$$= q^{n(n-1)/2-(2n+4)} > q^3 > 10,$$

a contradiction. Suppose that $n = 6$ and $q \geq 4$. If $p = 3$, then since $q^i - 1 > q^{i-1}(q-1)$ for $2 \leq i \leq 6$,

$$|G:H|_{3'} \geq \frac{(q^6-1)\cdots(q^2-1)/(6,q-1)}{q^{16}} > \frac{q^{5+\cdots+1}(q-1)^4}{q^{16}} = \frac{(q-1)^4}{q} > 10,$$

a contradiction. If $p \neq 3$ then $3 \nmid q^3 + \varepsilon$ for some $\varepsilon = 1$ or -1, so

$$|G:H|_{3'} \geq \frac{q^{15}(q^3+\varepsilon)}{q^{16}} = \frac{(q^3+\varepsilon)}{q} > 10,$$

a contradiction. Thus $n \leq 5$, or $n = 6$ and $q \leq 3$. By [19, 5.1.1–5.1.3 and 5.2.1–5.2.2],

G	$soc(H)$ $(\lvert soc(H)\rvert)$
$L_2(q)$	A_5 $(2^2\cdot 3\cdot 5)$
$L_3(q)$	A_6 $(2^3\cdot 3^2\cdot 5)$, $L_2(7)$ $(2^3\cdot 3\cdot 7)$
$L_4(q)$	A_7 $(2^3\cdot 3^2\cdot 5\cdot 7)$, $L_2(7)$ $(2^3\cdot 3\cdot 7)$, $U_4(2)$ $(2^6\cdot 3^4\cdot 5)$
$L_5(q)$	$PSp_4(3)$ $(2^6\cdot 3^4\cdot 5)$, $L_2(11)$ $(2^2\cdot 3\cdot 5\cdot 11)$, M_{11} $(2^4\cdot 3^2\cdot 5\cdot 11)$
$L_6(3)$	A_6 $(2^3 3^2 5)$, $L_2(11)$ $(2^2 3\cdot 5\cdot 11)$, $L_3(3)$ $(2^4 3^3 13)$, M_{12} $(2^6 3^3 5\cdot 11)$

If $(G, soc(H)) = (L_2(q), A_5)$, then since $\pi(|H|) = \{2,3,5\}$, we have $\pi(|G|) = \{2,3,5\}$. It follows from Lemma 2.3(1) that $G = L_2(9) \cong A_6$. However $3 \nmid |G:H|$, a contradiction. If $G = L_3(q)$ or $L_4(q)$, then $\pi(|H|) \subseteq \{2,3,5,7\}$ and $7^2 \nmid |H|$. Thus $\pi(|G|) \subseteq \{2,3,5,7\}$ and $7^2 \nmid |G|$. By Lemma 2.3 and [3], no such G exists such that $|G:H| \mid 3^r 10$. Suppose $G = L_5(q)$. Then $(q^5-1)(q^4-1)(q^3-1) \mid |G|$. If $q^3 = 2^6$ then $G = L_5(4)$, so $31 = 2^5 - 1 \mid |G:H|$, a contradiction. If $q^3 \neq 2^6$, then by Lemma 2.1, $|G|$ has at least three different prime divisors greater than 3. It follows that $|G:H|$ has a prime divisor greater than 5, a contradiction. If $G = L_6(3)$ then $7 \mid |G|$, so $7 \mid |G:H|$, a contradiction.

For the other cases, similar arguments deduce that the lemma is true. □

Now suppose that G is a classical simple group and that H is a maximal subgroup of G in collections $\mathscr{C}_1$–$\mathscr{C}_8$, which has index $3^r l$.

Lemma 3.10. *If $H \in \mathscr{C}_1$, then Theorem* 1.1 *holds.*

Proof. Assume $G = L_n(q)$. By [14, 4.1.17],

$$H = [q^{m(n-m)}]\cdot \left[\frac{a}{(q-1,n)}\right]\cdot (L_m(q) \times L_{n-m}(q))\cdot [b],$$

where $1 \le m \le n-1$, $a = |\{(\lambda_1, \lambda_2) \in GF(q^2)^* \times GF(q^2)^* \mid \lambda_i^{q-1} = 1,\ \lambda_1^m \lambda_2^{n-m} = 1\}|$ and $b = (q-1)(q-1,m)(q-1,n-m)/a$. Thus any prime divisor of $|H|$ divides q or $q^i - 1$ for $i < n$. First suppose $n \ge 3$. If $q^n = 2^6$, then either $G = L_3(4)$ and $H = 2^4{\cdot}L_2(4)$, or $G = L_6(2)$ and $H = [2^5]{\cdot}GL_5(2)$, $[2^6]{\cdot}(SL_2(2) \times SL_4(2))$ or $[2^9]{\cdot}(SL_3(2) \times SL_3(2))$. It is easy to show that 7 or 31 divides $|G : H|$, a contradiction. If $q^n \ne 2^6$ then $q^n - 1 \mid |G|$ and, by Lemma 2.1, there is a prime $k \mid q^n - 1$ such that $k > n$ and $k \nmid q^i - 1$ for all $i < n$. Thus $k \mid |G : H|$, so $k \le 5$ and $n \le 4$. If $n = 3$ then

$$H = [q^2]\cdot \left[\frac{q-1}{(q-1,3)}\right]\cdot L_2(q)\cdot [(q-1,2)].$$

Thus $|G : H| = (q^3-1)/(q-1) = q^2+q+1$. By Lemma 2.1, either $p^{3e} = 2^6$, or there is a prime $k \mid p^{3e} - 1$ such that $k > 3e \ge 3$ and $k \nmid p^{2e} - 1$. If $p^{3e} = 2^6$ then $G = L_3(4)$ and $|G : H| = 21$, a contradiction. If $p^{3e} \ne 2^6$ then $k \mid |G : H|$, so $k = 5$ and $e = 1$. Now $5 \mid p^4 - 1$ and $5 \nmid p^2 - 1$, so $5 \mid p^2 + 1$, contrary to $5 \mid p^2 + p + 1$. If $n = 4$ then

$$|G : H| = \begin{cases} (q^2+1)(q+1) & \text{if } m = 1, \\ (q^2+1)(q^2+q+1) & \text{if } m = 2. \end{cases}$$

If $|G : H| = (q^2+1)(q+1)$, then since $4 \nmid |G : H|$ and $(q^2+1, q+1) = 1$ or 2, we have q^2+1 and $q+1$ are odd, and so $q^2+1 \mid 5$ or $q+1 \mid 5$. If $q^2+1 \mid 5$ then $q+1 = 3^r$, so $q = 2$. That is, $G = L_4(2) \cong A_8$ and $H = 2^3{\cdot}L_3(2)$. If $q+1 \mid 5$ then $q^2+1 = 3^r$, which is not possible. If $|G : H| = (q^2+1)(q^2+q+1)$, then since $(q^2+1, q^2+q+1) = 1$, either $q^2+1 \mid 10$ and $q^2+q+1 = 3^r$, or $q^2+q+1 \mid 10$ and $q^2+1 = 3^r$, both of which are not possible. Now suppose $n = 2$. Then

$$H = [q]\cdot \left[\frac{q-1}{(q-1,2)}\right] \quad \text{and} \quad |G : H| = q+1.$$

If $q = p$ then clearly $p+1$ is even, which is listed in the lemma. If $q = p^e > p$, then by Lemma 2.1, either $p^e = 2^3$ or there is a prime $k \mid p^e + 1$ such that $k > 2e \ge 4$. Thus either $p^e = 2^3$, or $e = 2$ and $5 = k \mid |G : H|$. If $p^e = 8$ then $G = L_2(8)$ and $H = 2^3{\cdot}7$; if $e = 2$ then either $p^e = 4$ or $p^e + 1$ is even. So either $G = L_2(4)$ and $|G : H| = 5$, or $|G : H| = p^e + 1 = 10{\cdot}3^r$.

Assume $G = U_n(q)$, $n \geq 3$. Then $q^n - (-1)^n \big| |G|$. If n is odd, then $q^n + 1 \big| |G|$. Note that $q^n \neq 2^3$. By Lemma 2.1, there is a prime $k \big| q^n + 1$ such that $k > 2n$ and $k \not| q^i - 1$ for all $i < 2n$. By [14, 4.1.4 and 4.1.18], any prime divisor of $|H|$ divides q or $q^i - 1$ for $i \leq 2n - 2$. Thus $k > 5$ is a prime divisor of $|G : H|$, a contradiction. If n is even, then $q^{n-1} + 1 \big| |G|$. If $q^{n-1} = 2^3$ then $G = U_4(2)$ and $H = 2^4{\cdot}A_5$ by [14, 4.1.18]. Suppose $q^{n-1} \neq 2^3$. By Lemma 2.1, there is a prime $k \big| q^{n-1} + 1$ such that $k > 2(n-1) > 5$ and $k \not| q^i - 1$ for all $i < 2(n-1)$. Again by [14, 4.1.4 and 4.1.18], any prime divisor of $|H|$ divides q or $q^i - 1$ for $i < 2(n-1)$. So $k \big| |G : H|$, a contradiction.

Assume $G = PSp_n(q)$, $n \geq 4$. Then $q^n - 1 \big| |G|$ and by [14, 4.1.3 and 4.1.19],

$$H = \begin{cases} (2, q-1) \cdot (PSp_m(q) \times PSp_{n-m}(q)) \ (2 \leq m < n/2) \text{ or} \\ [q^a]{\cdot}(q-1) \cdot (PGL_m(q) \times PSp_{n-2m}(q)) \ (1 \leq m \leq n/2), \end{cases}$$

where $a = m/2 - 3m^2/2 + mn$. Thus any prime divisor of $|H|$ divides q or $q^i - 1$ for $i < n$. If $q^n = 2^6$, then $G = Sp_6(2)$, $H = 2^6{\cdot}L_3(2)$ and $|G : H| = 3^3 5$. Suppose $q^n \neq 2^6$. By Lemma 2.1, there is a prime $k \big| q^n - 1$ such that $k > n$ and $k \not| q^i - 1$ for all $i < n$. Thus $k \not| |H|$ and $k \big| |G : H|$, so $k \leq 5$ and $n = 4$. It follows that $H = [q^a]{\cdot}(q - 1){\cdot}(PGL_m(q) \times PSp_{4-2m}(q))$ and $m = 1$ or 2. If $m = 1$ then $H = [q^3]{\cdot}(q-1){\cdot}PSp_2(q)$. Thus $|G : H| = (q^2 + 1)(q + 1) = 3^r l$. Since $4 \not| |G : H|$ and $(q^2 + 1, q + 1) = 1$ or 2, q is even. So either $q^2 + 1 = 3^r$ and $q + 1 \big| 5$, or $q + 1 = 3^r$ and $q^2 + 1 \big| 5$. Hence $G = PSp_4(2) \cong A_6$ and $H = S_4$. If $m = 2$ then $H = [q^3]{\cdot}(q-1){\cdot}PGL_2(q)$. Similarly arguments can show $|G : H| \not| 3^r 10$, a contradiction.

Assume $G = P\Omega_n^{\pm}(q)$. Then $n \geq 8$ and $q^{n-2} - 1 \big| |G|$. If $q^{n-2} = 2^6$ then $G = P\Omega_8^{\pm}(2)$. By [14, Section 4.1] and [3], $H = 2^6{\cdot}A_8$ and $|G : H| = 3^3{\cdot}5$. Suppose $q^{n-2} \neq 2^6$. By Lemma 2.1, there is a prime $k \big| q^{n-2} - 1$ such that $k > n - 2 \geq 5$ and $k \not| q^i - 1$ for all $i < n - 2$. However, by [14, 4.1.6–.7 and 4.1.20], one of the following holds:

(a) $G = P\Omega_n^{\pm}(q)$ and $H = \Omega_{n-1}(q)$;

(b) $G = P\Omega_n^{\pm}(q)$ and $H = Sp_{n-2}(q)$;

(c) any prime divisor of $|H|$ divides q or $q^i - 1$ for $i < n - 2$.

If (c) holds then $k \big| |G : H|$, a contradiction. If (a) holds then $|G|/|H| = (2/(4, q^m - 1))$ $q^{m-1}(q^m \pm 1)$, where $m = n/2 \geq 4$. Thus $p = 3$ and $|G : H|_{3'} \geq (q^m - 1)/2 > 10$, a contradiction. Similarly, case (b) cannot hold.

Assume $G = P\Omega_n(q)$, n odd. Then arguing as for the case $G = P\Omega_{2m}^{\pm}(q)$ in the previous paragraph, no H satisfies our requirement. □

If H lies in one of collections $\mathscr{C}_2$–$\mathscr{C}_8$, then similar arguments can prove Theorem 1.1.

4. Proofs of Theorems 1.2 and 1.3

Proof of Theorem 1.2. Let $soc(G) = T_1 \times \cdots \times T_k$ for $k \geq 1$. Since G is a primitive group, G satisfies the O'Nan–Scott theorem, see [18]. If T_i is abelian then part (i) holds.

Assume that T_i is nonabelian and $k > 1$. Then clearly G is neither of type II (a), nor of type III (b) (ii), nor of type III (c) (in terms of [18]). If G is a group of type III (b) (i), then since 2^2 and 5^2 do not divide n, T_i has a maximal subgroup H_i of index 3^s for some integer $s \mid r$. By Theorem 1.1, part (ii) holds.

Finally, assume that $soc(G) = T$ is a nonabelian simple group and that H is a maximal subgroup of G of index $3^r l$. Then $T \lhd G \leq \mathrm{Aut}(T)$. Since $|HT| = |H||T|/|H \cap T|$, we have $|T|/|H \cap T| = |HT|/|H|$, which divides $|G|/|H|$. So $H \cap T$ is a subgroup of T of index $3^s{\cdot}l'$ where $s \leq r$ and $l' \mid 10$. By Table 1, it is easy to obtain Table 2. □

Now we are going to prove Gardiner and Praeger's conjecture by checking the groups in Table 2. First we prove a simple lemma. Let G be a transitive permutation group of degree n on a set Ω and let $H = G_\alpha$ where $\alpha \in \Omega$. Suppose $n = 3^r l$ for some $l \mid 10$. For $g \in G$, write g as a product of disjoint cycles $(a_1a_2\ldots)(b_1b_2\ldots)\ldots(z_1z_2\ldots)$. Select the first point, the third, fifth, etc. from each cycle, but where a cycle has odd length omit the last point. These points form a set Δ satisfying $\Delta \cap \Delta^g = \emptyset$. Denote this Δ by $\Delta(g)$.

Lemma 4.1. *If $5 \mid n$ and $5 \nmid |H|$ then G does not have the property* BM$(n/3)$.

Proof. Now G has an element g of order 5 which fixes no point of Ω. Thus $\Delta(g)$ has length $2{\cdot}(n/5)$. Since $2n/5 > n/3$, G does not have the property BM$(n/3)$. □

Proof of Theorem 1.3. We only need to prove that no group in Table 2 has the property BM$(n/3)$, except for the case $G = A_5$. By Lemma 4.1, if $(G,H) = (A_6,S_4)$, $(A_7,L_2(7))$, $(A_8,2^3{\cdot}L_3(2))$, $(L_2(8),2^3{\cdot}7)$, $(U_4(2),2{\cdot}(A_4 \times A_4){\cdot}2)$ or $(Sp_6(2),2^6{\cdot}L_3(2))$, then G does not have the property BM$(n/3)$. If $(G,H) = (U_4(3),L_3(4))$, then G is of rank three and has nontrivial suborbits Ω_1 and Ω_2 of sizes 56 and 105. Let g be an element of G of order 7. Since $7^2 \nmid |G|$ and $7 \mid 56$ and 105, g has no fixed point in Ω_1 and Ω_2. Thus g fixes only the point α of Ω. Hence $\Delta(g)$ from g has length $3.\frac{161}{7} > \frac{162}{3}$, so G does not have the property BM$(\frac{162}{3})$. If $(G,H) = (A_{10},S_8)$ or $(P\Omega_8^+(2),2^6{\cdot}A_8)$, then G contains at least two conjugacy classes of subgroups of order 5. Since all subgroups of H of order 5 are conjugate and all point-stabilizers in G are conjugate, we know that all subgroups of order 5 which fix at least one point of Ω are conjugate. Thus G has an element g of order 5 which fixes no point. So if $G = A_{10}$ then $\Delta(g)$ has length $2{\cdot}\frac{45}{5} > \frac{45}{3}$; if $G = P\Omega_8^+(2)$ then $\Delta(g)$ has length $2{\cdot}\frac{135}{5} > \frac{135}{3}$. Hence G does not have the property BM$(n/3)$. If $G = U_4(2)$ and $H = 2^4{\cdot}A_5$, then H fixes α and has nontrivial orbits Ω_1 and Ω_2 on $\Omega\backslash\{\alpha\}$ of sizes 10 and 16 respectively. Let g be an element of H

of order 5. Then g has no fixed point in Ω_1 since $H|_{\Omega_1}$ has the point-stabilizer isomorphic to $2^4{\cdot}D_6$ with no element of order 5. Note that $H|_{\Omega_2}$ is 2-transitive on Ω_2. Thus $H_\beta \cong A_5$ is transitive on $\Omega_2\backslash\{\beta\}$ for some $\beta \in \Omega_2$. Let $\gamma \in \Omega_2\backslash\{\beta\}$. Then $H_{\beta\gamma} = A_4$ is of order coprime to 5. Thus g fixes no point in $\Omega_2\backslash\{\beta\}$. Hence g fixes no point in $\Omega\backslash\{\alpha,\beta\} = \Omega_1 \cup \Omega_2\backslash\{\beta\}$. So we can construct a $\Delta(g)$ from g, which has length $2{\cdot}\frac{10+16-1}{5} = 10 > \frac{27}{3}$. So G does not have the property BM($\frac{27}{3}$).

Now assume $G = A_c$ and $H = A_{c-1}$. If $c = 6$, then G has an element g of order 4 which fixes no point in Ω. Thus we can get a set $\Delta(g)$ of size 3 such that $\Delta(g)\cap\Delta(g)^g = \emptyset$, so G does not have the property BM(2). Suppose $c > 6$. Since $4 \nmid c$, $4 \mid c-i$ for some $i = 1, 2$ or 3. Thus G has an involution g which exactly fixes i points of Ω. Hence $\Delta(g)$ from g has length $(c-i)/2$. Note that now $c \geq 9$ and if $c = 9$ then $i = 1$. We have $(c-i)/2 > c/3$, so G does not have the property BM($4c/3$).

Assume $G = L_2(q)$ and $H = [q]{\cdot}Z_{(q-1)/2}$, where $q = p$ or p^2, and q is odd. Then $n = |G : H| = q + 1$. Since $3 \mid q+1$, we have $p \neq 3$. Since G is 2-transitive on Ω, H is transitive on $\Omega\backslash\{\alpha\}$. Let $\beta \in \Omega\backslash\{\alpha\}$. Then $H_{\alpha\beta} = Z_{(q-1)/2}$ and $(q, |H_{\alpha\beta}|) = 1$. Thus H_α has an element g of order p which fixes no point in $\Omega\backslash\{\alpha\}$. Let $\Delta(g)$ be the set that came from g. Then

$$|\Delta(g)| = \frac{p-1}{2}\cdot\frac{n-1}{p} > \frac{n}{3}.$$

Thus G does not have the property BM($n/3$). This completes the proof of Theorem 1.3. □

Final remark. While this paper was in preparation, the author was told that A. Mann and C.E. Praeger proved Gardiner and Praeger's conjecture. Our work is independent of, and the methods used here are different to, theirs.

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